Constructions with a compass

Lionel Fourquaux

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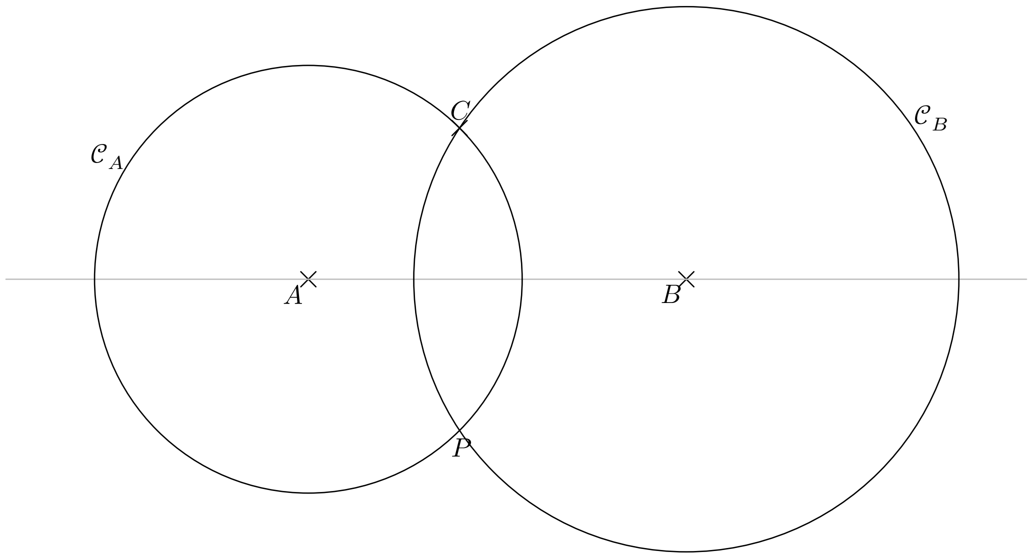
1 Axial symmetry

Initial data:

- a line (AB);
- a point $C \notin (AB)$.

Constructs: the image of C by the axial symmetry of axis (AB).



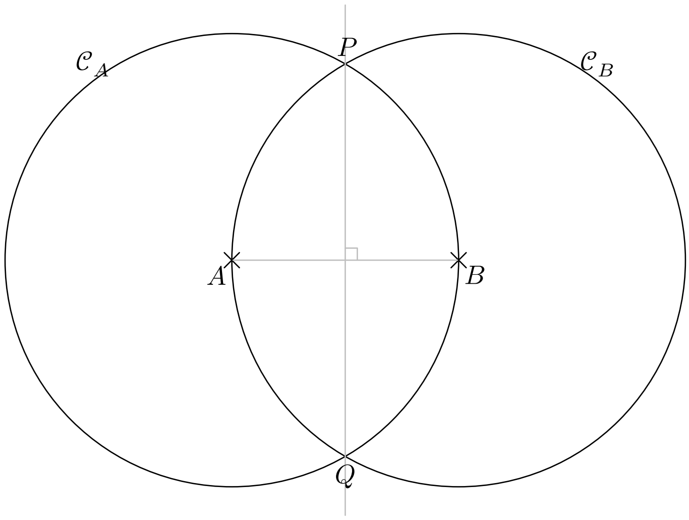
Let $\mathcal{C}_A$ be the circle of centre A going through C. Let $\mathcal{C}_B$ be the circle of centre B going through C. Let P be the intersection point of $\mathcal{C}_A$ and $\mathcal{C}_B$ that is not C. Then P is the image of C by the axial symmetry of axis (AB).

Proof. Since $AC = AP$ and $BC = BP$, the line (AB) is the perpendicular bisector of the segment $[PC]$. □

2 Perpendicular bisector

Initial data: a segment $[AB]$.

Constructs: the perpendicular bisector of the segment $[AB]$.



Let $\mathcal{C}_A$ be the circle of centre A going through B. Let $\mathcal{C}_B$ be the circle of centre B going through A. Let P and Q be the intersection points of $\mathcal{C}_A$ and $\mathcal{C}_B$. Then (PQ) is the perpendicular bisector of the segment $[AB]$.

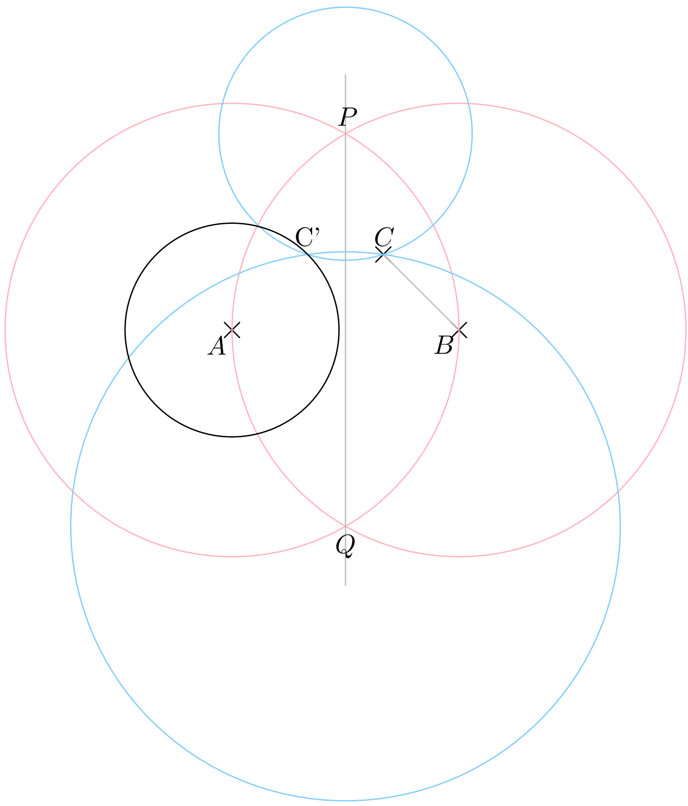
Proof. Since $PA = AB = PB$ and $QA = AB = QB$, the points P and Q are on the perpendicular bisector. □

3 Circle with given radius

Initial data: three points A, B and C.

Constructs: the circle of center A and radius BC.

Uses: 1, 2.



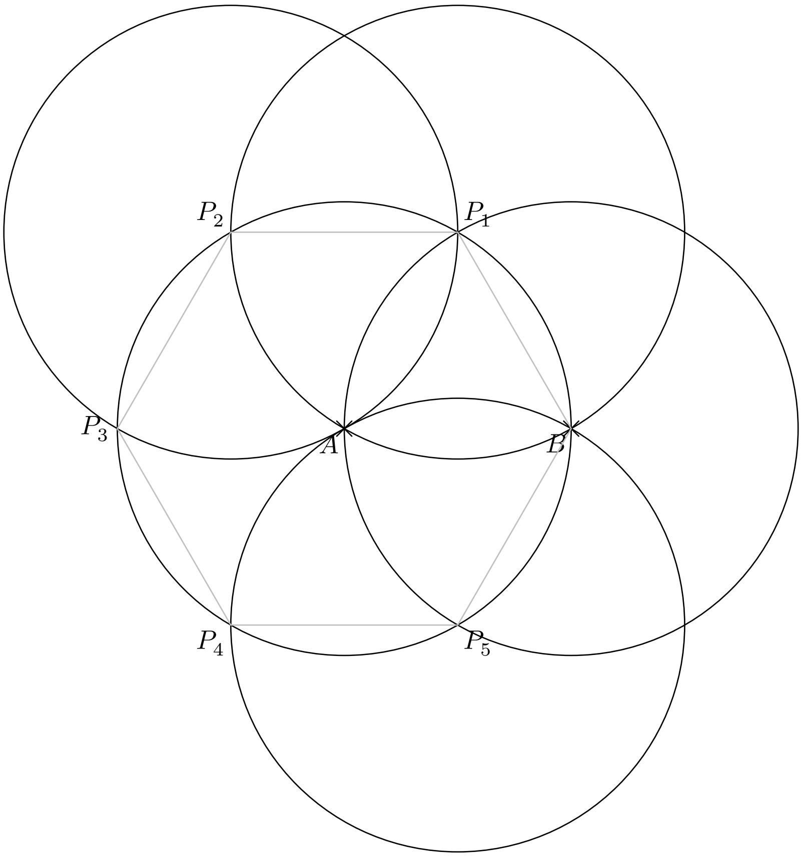
Using 2, construct the perpendicular bisector of the segment $[AB]$. Then, using 1, construct the image C' of C by the axial symmetry of axis (PQ). Finally, draw the circle of center A going through C'.

Proof. The axial symmetry sends B to A and C to C', thus $AC' = BC$. □

4 Regular hexagon

Initial data: two points A and B.

Constructs: the regular hexagon of center A and vertex B.



Draw the circle of centre B going through A and the circle of centre A going through B. Let P_1 and P_5 be the intersections of these circles. Draw the circle of centre P_1 going through A and let P_2 be the intersection of the circles of centre A and P_1 that is not B. Draw the circle of centre P_5 going through A and let P_4 be the intersection of the circles of centre A and P_5 that is not B. Finally draw the circle of centre P_2 going through A and let P_2 be the intersection of the circles of centre A and P_2 that is not P_1. Then $BP_1P_2P_3P_4P_5$ is a regular hexagon.

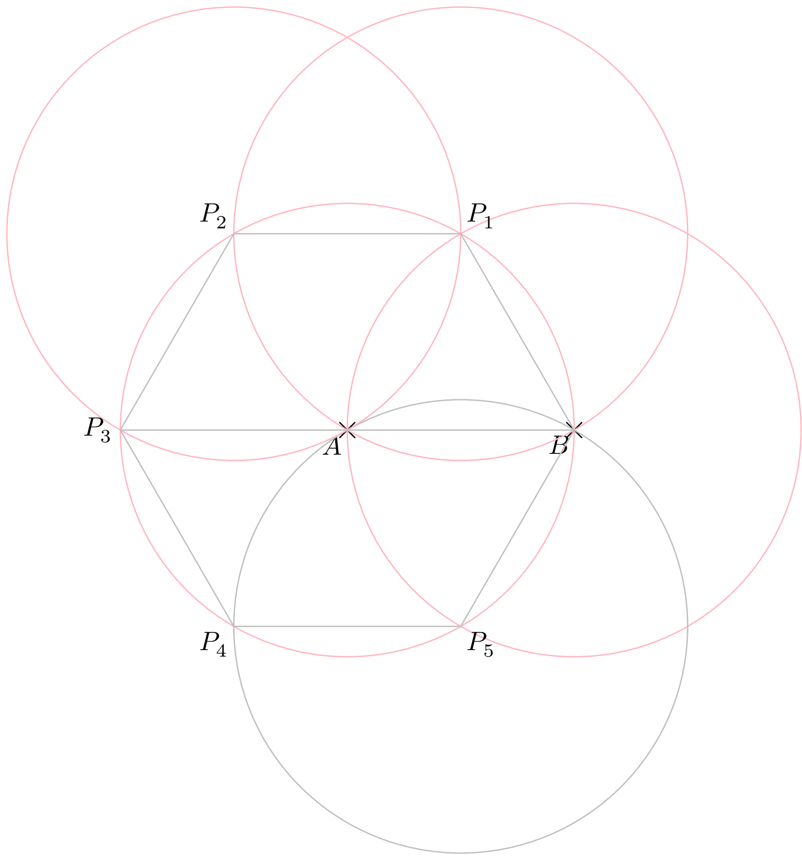
Proof. The triangles ABP_1, AP_1P_2, AP_2P_3, AP_4P_5 and AP_5B are equilateral. □

5 Central inversion

Initial data: two points A and B.

Constructs: the image of B by the central inversion of center A.

Uses: 4.



Using 4, construct the regular hexagon $BP_1P_2P_3P_4P_5$ of centre A. Then P_3 is the image of B by the central inversion of centre A. (Actually, P_4 is not needed, so the circle of center P_5 needs not be drawn.)

Proof. Since the hexagon is regular: $\widehat{BAP_3} = 3\frac{\pi}{3} = \pi$. □

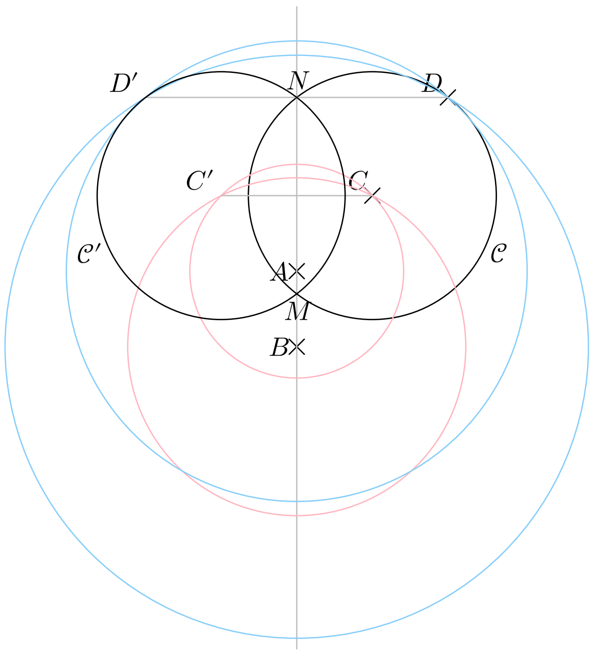
6 Intersection of a line and a circle

Initial data:

- a line (AB);
- a circle $\mathcal{C}$ of **known** centre $C \notin (AB)$;
- a point $D \in \mathcal{C}$.

Constructs: the intersection points of $\mathcal{C}$ and (AB).

Uses: 1.



Using 1, construct the images C' and D' of C and D by the axial symmetry of axis (AB). Draw the circle $\mathcal{C}'$ of centre C' going through F'. The intersection of $\mathcal{C}$ and (AB) are the intersections M and N of $\mathcal{C}$ and $\mathcal{C}'$.

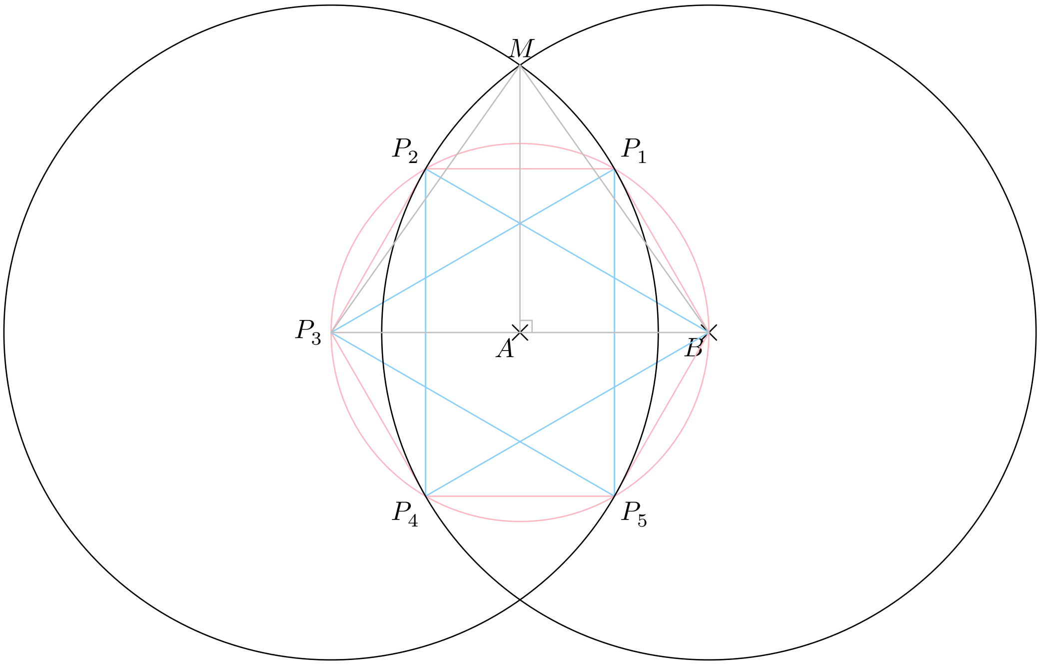
Proof. The circle $\mathcal{C}'$ is the image of $\mathcal{C}$ by the axial symmetry of axis (AB), thus the intersection points of $\mathcal{C}$ and (AB), which are fixed by this symmetry, are also on $\mathcal{C}'$. □

7 Square root of 2

Initial data: a segment $[AB]$.

Constructs: a segment of length $\sqrt{2}\,AB$.

Uses: 4.



Using 4, construct the regular hexagon $BP_1P_2P_3P_4P_5$ of centre A. Draw the circle of centre B going through P_2 and the circle of centre P_3 going through P_1. Let M be one intersection of these two circles. Then $AM = \sqrt{2}\,AB$.

Proof. The triangles BP_2P_4 and $P_1P_3P_5$ are equilateral because the hexagon is regular. Thus,

$$MB = MP_3 = BP_2 = P_3P_1 = \sqrt{3}\,AB.$$

The triangle BMP_3 is isosceles of apex M. Moreover, A is the midpoint of MP_3, hence the triangle MAB is rectangle at A. By Pythagoras' theorem,

$$AM = \sqrt{MB^2 - AB^2} = \sqrt{5AB^2 - AB^2} = \sqrt{2}\,AB.$$

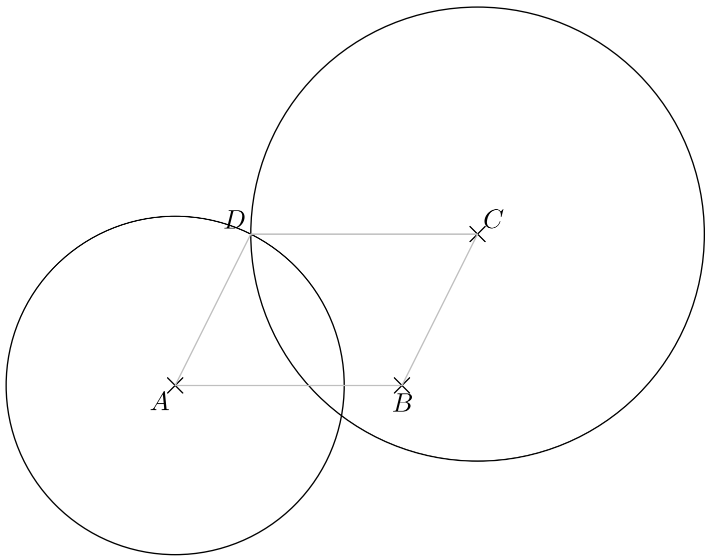
□

8 Parallelogram

Initial data: three points A, B and C, not aligned.

Constructs: a parallelogram $ABCD$.

Uses: 3.



Using 3, construct the circle of centre A and radius BC, and the circle of centre C and radius AB. Then D is one of the intersections of these two circles.

Proof. Here, D is the image of A by the translation of vector $\overrightarrow{BC}$, and the image of C by the translation of vector $\overrightarrow{BA}$. (They are the same by Chasles' relation.) Thus, $AD = BC$ and $CD = AB$. □

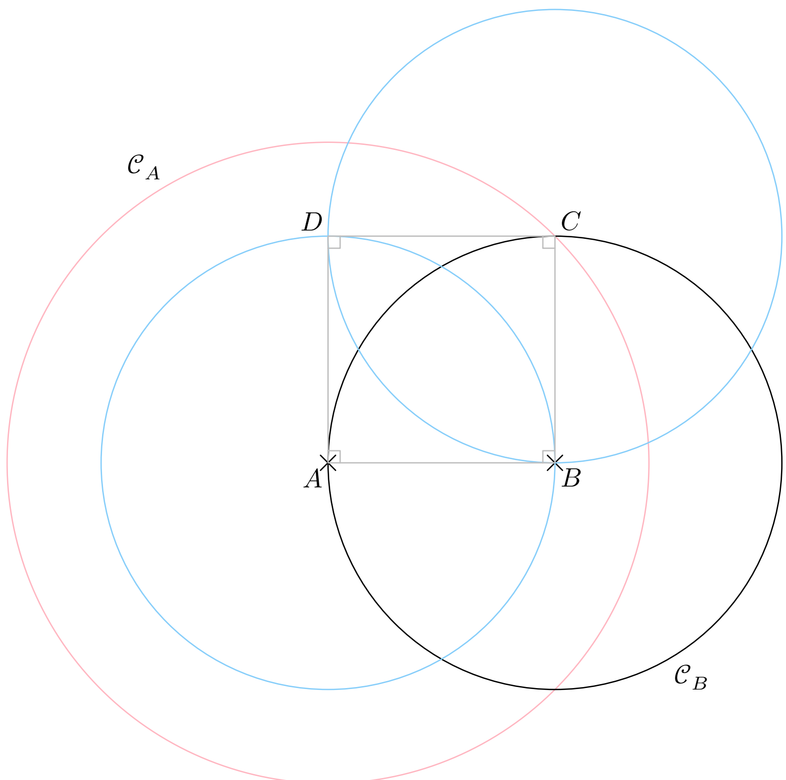
An alternative, more complicated, construction would be to construct the midpoint I of the segment $[AC]$ using 10, then D as the image of B by the central inversion of centre I by 5.

9 Square

Initial data: a segment $[AB]$.

Constructs: a square $ABCD$.

Uses: 1, 3, 7.



Using 7 and 3, draw the circle $\mathcal{C}_A$ of centre A and radius $\sqrt{2}\,AB$. Draw the circle $\mathcal{C}_B$ of centre B going through A. Let C be one of the intersections of these two circles. By 1, construct the image D of B by the axial symmetry of axis (AC). Then $ABCD$ is a square.

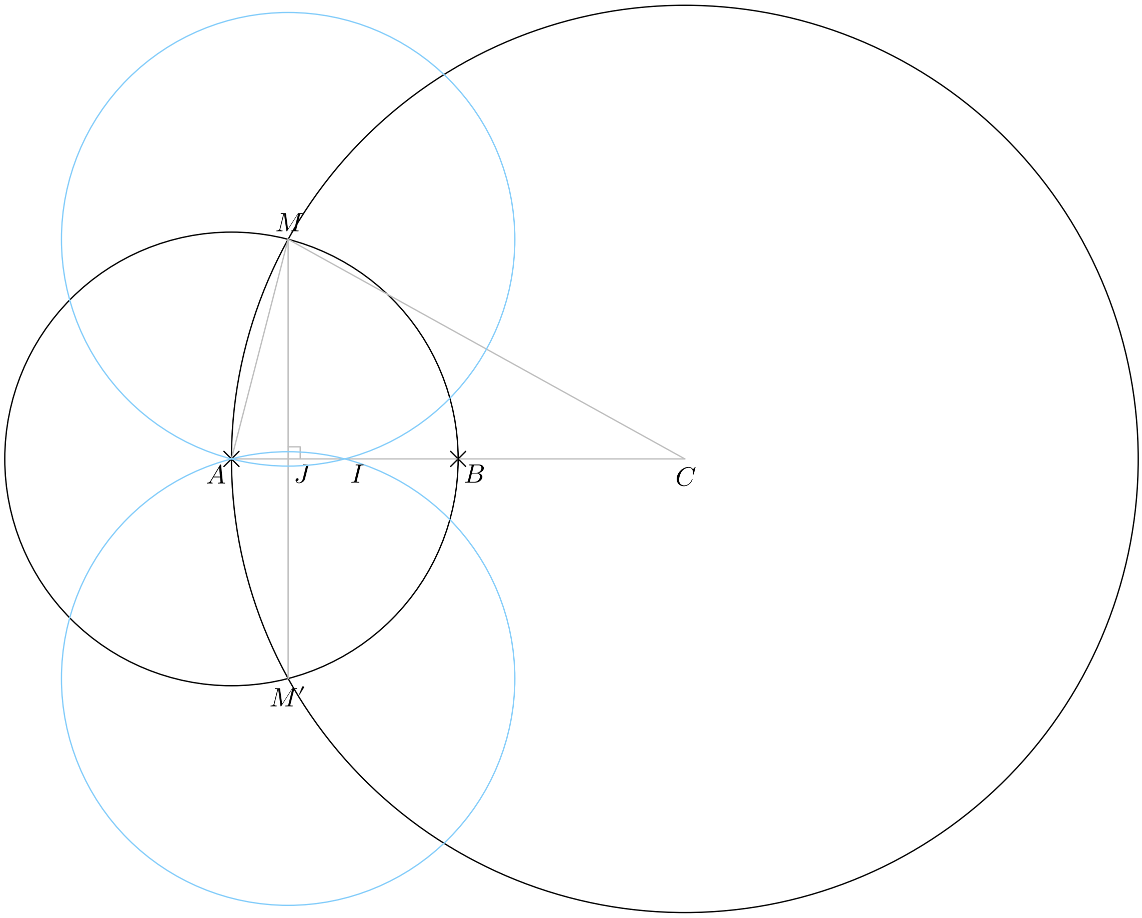
Proof. By Pythagoras' theorem, the diagonal of such a square has length equal to $\sqrt{2}\,AB$. □

10 Midpoint

Initial data: a segment $[AB]$.

Constructs: the midpoint I of $[AB]$.

Uses: 1, 5.



Using 5, construct the image C of A by the central inversion of centre B. Draw the circle of centre A going through B and the circle of centre C going through A. Let M and M' be the intersections of these two circles. Then I is the image of A by the axial symmetry of axis (MM'), which can be constructed using 1.

Proof. Let J be the orthogonal projection of M on the line (AB). (This point is not part of the construction.) Then, by Pythagoras' theorem,

$$AM^2 = AJ^2 2 + MJ^2 \qquad \text{and} \qquad CM^2 = JC^2 + MJ^2.$$

Since $AM = AB$ and $CM = CA = 2AB$, this implies

$$(JC - AJ)(JC + AJ) = JC^2 - AJ^2 = CM^2 - AM^2 = 3AB^2.$$

On the other hand, $JC + AJ = AC = 2AB$. Thus,

$$JC - AJ = \frac{3}{2}AB \qquad \text{and} \qquad JC + AJ = 2AB.$$

Hence, $AJ = \frac{1}{4}AB$, and J is the midpoint of $[AI]$. □

11 Orthogonal projection

Initial data:

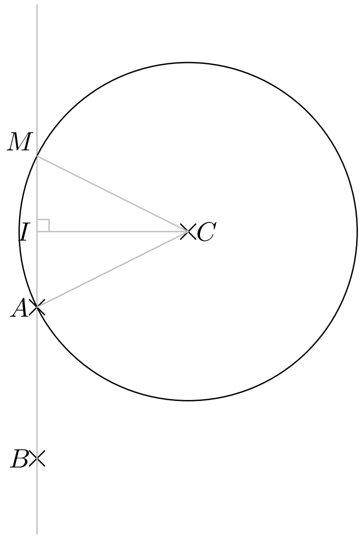
- a line (AB);
- a point C.

Constructs: the orthogonal projection of C on (AB).

Uses: 6, 10.

We can assume:

- that $C \notin (AB)$ (otherwise, the orthogonal projection of C is the point C itself);
- that the orthogonal projection of C is not A.



Draw the circle of centre C going through A. Using 6, construct the intersection point M of this circle with the line (AB) that is not A. Using 10, construct the midpoint I of the segment $[AM]$.

Proof. The triangle CAM is isosceles of apex C. □

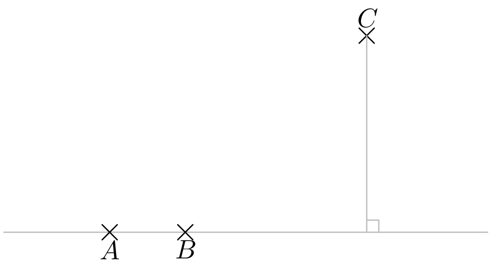
12 Abscissa of a point

Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(1, 0)$;
- a point C of coordinates (x, y).

Constructs: a point of coordinates $(x, 0)$.

Uses: 11.



Using 11, construct the orthogonal projection of C on the line (AB). It has coordinates $(x, 0)$.

Proof. Obvious. □

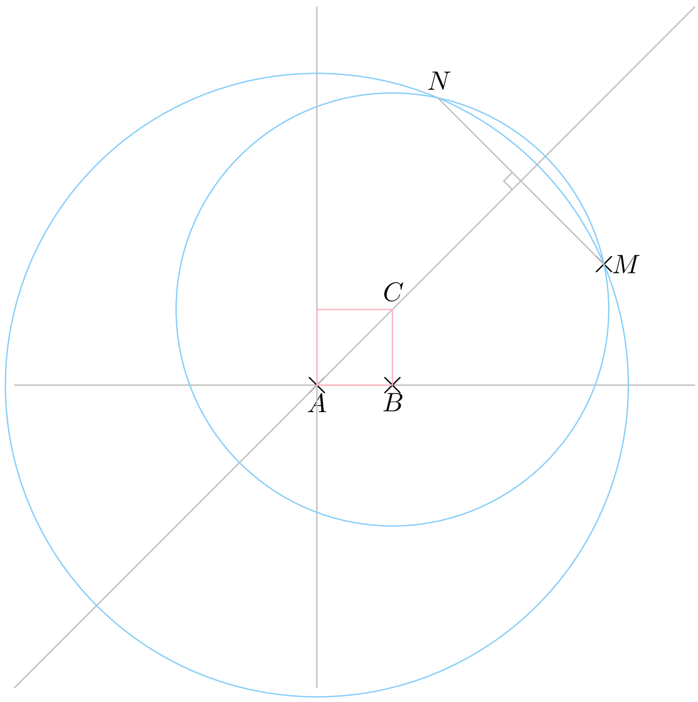
13 Swapping coordinates

Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(1, 0)$;
- a point M of coordinates (x, y).

Constructs: a point of coordinates (y, x).

Uses: 1, 9.



Using 9, construct C of coordinates $(1, 1)$. Then, use 1 to construct the image N of M by the axial symmetry of axis (AC). The point N has coordinates (y, x).

Proof. If N' is the point of coordinates (y, x), then $AM = AN'$ and $CM = CN'$. □

14 Ordinate of a point

Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(1, 0)$;
- a point C of coordinates (x, y).

Constructs: a point of coordinates $(y, 0)$.

Uses: 12, 13.

First, construct the point of coordinates (y, x) using 13, then use 12.

Proof. Obvious. □

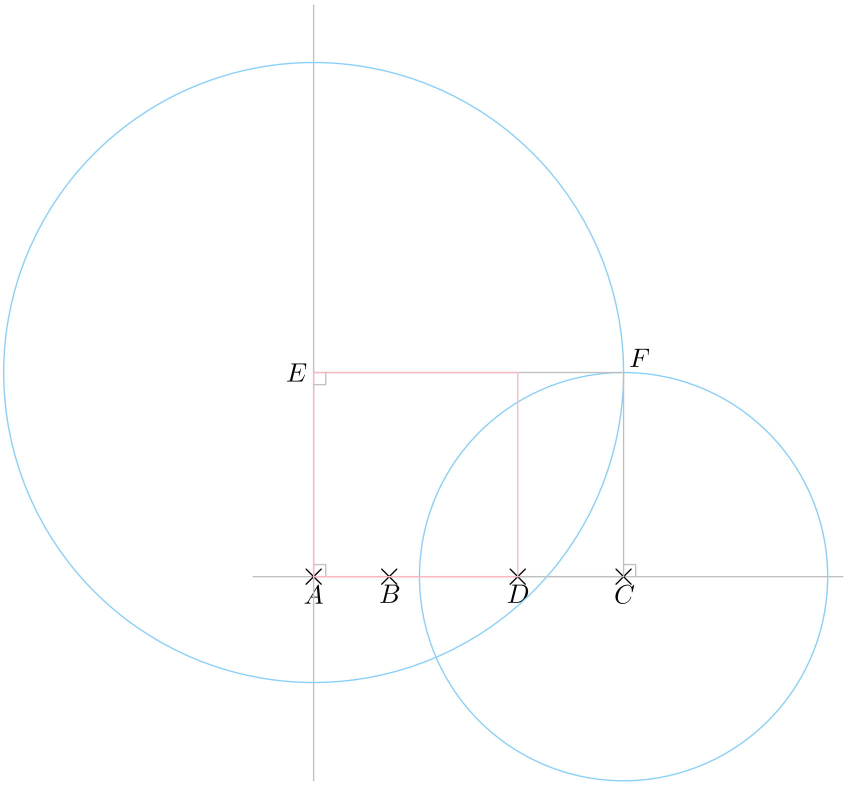
15 Point from coordinates

Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(1, 0)$;
- a point C of coordinates $(x, 0)$;
- a point D of coordinates $(y, 0)$.

Constructs: a point of coordinates (x, y).

Uses: 8, 9.



Use 9 to construct the point E of coordinates $(0, y)$, then use 8 to complete the rectangle $ACFE$. Then the point F has coordinates (x, y).

Proof. Obvious. □

16 Doubling

Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(x, 0)$.

Constructs: a point of coordinates $(2x, 0)$.

Uses: 5.

The image of A by the central inversion of centre B (constructed by 5) has coordinates $(2x, 0)$.

Proof. Obvious. □

17 Addition

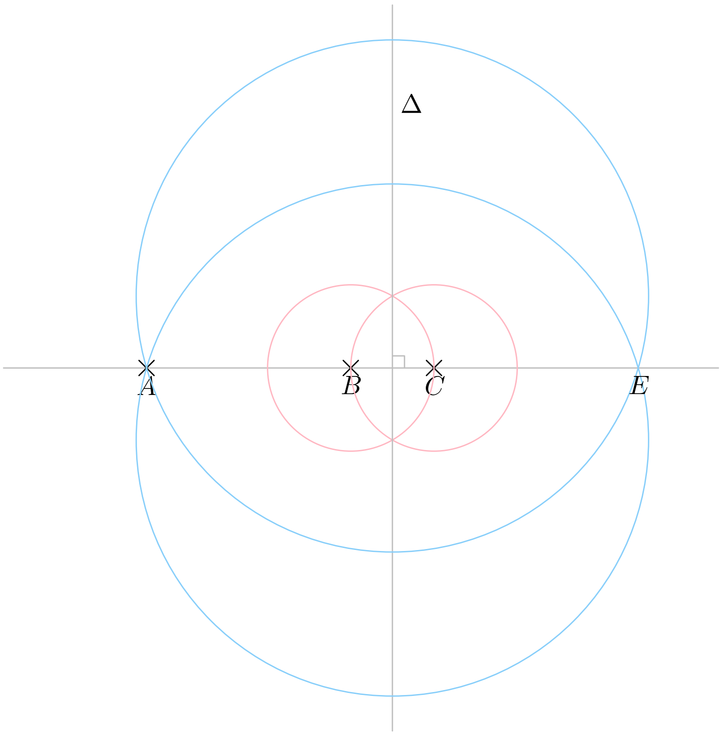
Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(x, 0)$;
- a point C of coordinates $(y, 0)$.

Constructs: a point of coordinates $(x + y, 0)$.

Uses: 1, 2.

We can assume $x \neq y$, otherwise this reduces to 16.



Using 2, construct the bisector Δ of the segment $[BC]$, then use 1 to construct the image D of A by the axial symmetry of axis Δ.

Proof. The line Δ is the set of points of ordinate $\frac{x+y}{2}$. □

18 Substraction

Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(x, 0)$;
- a point C of coordinates $(y, 0)$.

Constructs: a point of coordinates $(x - y, 0)$.

Uses: 5, 17.

Using 5, construct the image of C by the central inversion of centre A. Its coordinates are $(-y, 0)$. Then use 17.

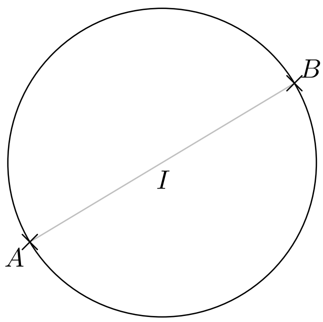
Proof. Obvious. □

19 Circle from diameter

Initial data: a segment $[AB]$.

Constructs: the circle of diameter $[AB]$.

Uses: 10.



Using 10, construct the midpoint I of the segment $[AB]$. Then draw the circle of centre I going through A.

Proof. Obvious. □

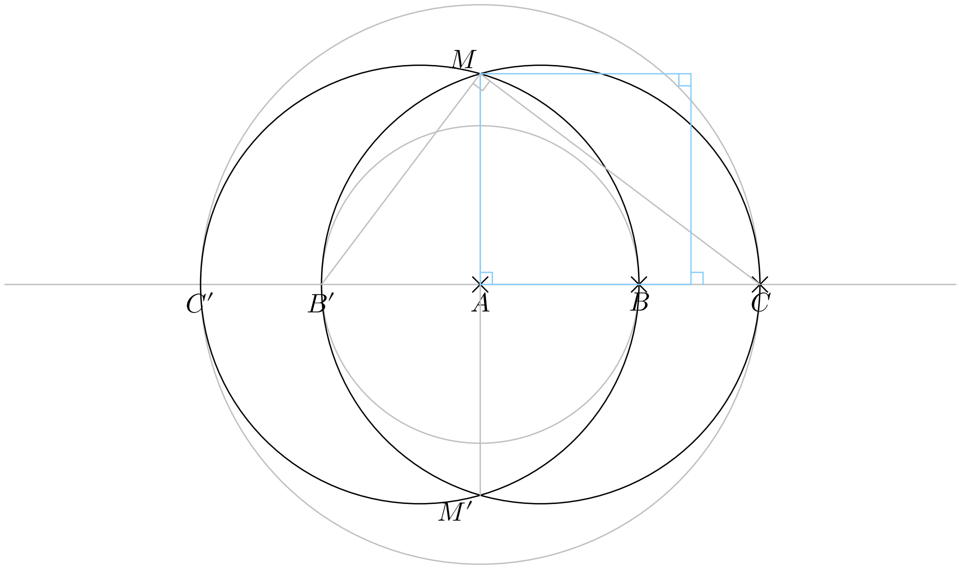
20 Square root of product

Initial data:

- a point A of coordinates $(0,0)$;
- a point B of coordinates $(x,0)$;
- a point C of coordinates $(y,0)$.

Constructs: a point of coordinates $(\sqrt{xy},0)$.

Uses: 5, 9, 19.



Using 5, construct the points B' of coordinates $(-x,0)$ and C' of coordinates $(-y,0)$. Then, using 19, draw the circles of diameters $[BC']$ and $[B'C]$. Let M and M' be the intersection points of these two circles. Their coordinates are $(0,\pm\sqrt{xy})$. Finally, use 9 to get a point of coordinates $(\sqrt{xy},0)$.

Proof. By symmetry of the drawing, the abscissas of the points A, M and M' are 0. By construction of M, the triangle $B'MC$ is rectangle at M.

Then, the triangles $B'MC$, $B'AM$ and MAC are similar. Thus,

$$\frac{AM}{x} = \frac{y}{AM},$$

from which we deduce $AM = \sqrt{xy}$. □

21 Square root

Initial data:

- a point of coordinates $(0,0)$;
- a point of coordinates $(1,0)$;
- a point of coordinates $(x,0)$.

Constructs: a point of coordinates $(\sqrt{x},0)$.

Uses: 20.

Use 20 with $y = 1$.

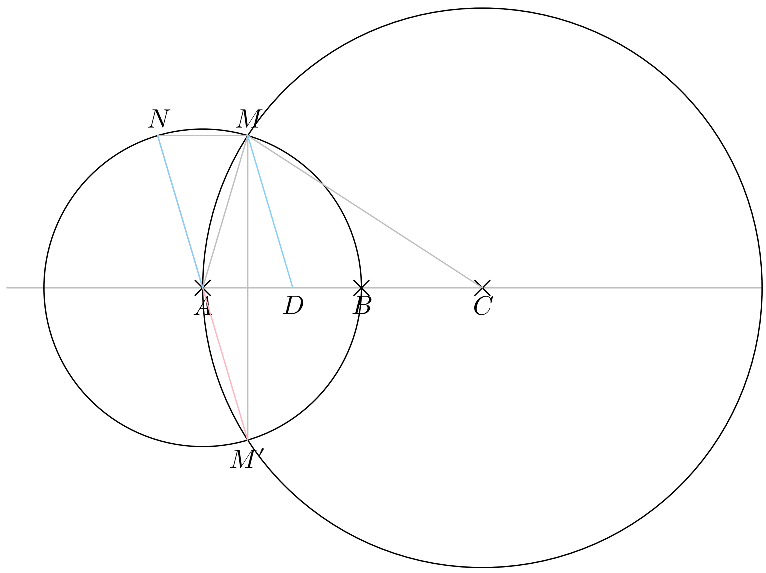
Proof. Obvious. □

22 Quotient of square

Initial data:

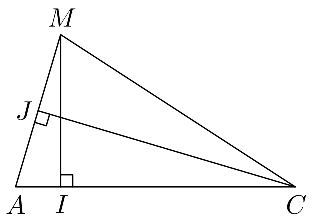
- a point A of coordinates $(0,0)$;
- a point B of coordinates $(x,0)$;
- a point C of coordinates $(y,0)$.

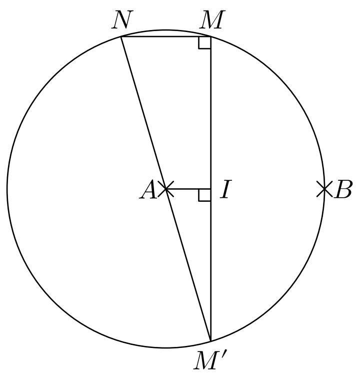
Constructs: a point of coordinates $\left(\frac{x^2}{y}, 0\right)$.
Uses: 5, 8, 16.



Draw the circle of centre A going through B, and the circle of centre C going through A. Use 5 to construct the image N of M' by the central inversion of centre A. Use 8 to complete the parallelogram $MNAD$. Then the point D has coordinates $\left(\frac{x^2}{y}, 0\right)$.

Proof. By construction, the triangle MAC is isosceles of apex C. Let I be the orthogonal projection of M on (AC), and J the orthogonal projection of C on (AM). Then I is the midpoint of $[MM']$ and J is the midpoint of $[AM]$.





The triangles MIA and CJA are similar, hence

$$\frac{AI}{x} = \frac{AI}{AM} = \frac{AJ}{AC} = \frac{x}{2y},$$

thus $AI = \frac{x^2}{2y}$.
Since $[M'N]$ is a diameter of the circle of centre A going through B, and M is on this circle, the triangle MNM' is rectangle at M. Since I is the orthogonal projection of M on (AC), the lines (NM) and (AI) are parallel. Thales' theorem applies, and since A is the midpoint of $[M'N]$, this implies:

$$AD = NM = 2AI = \frac{x^2}{y}.$$

□

This construction assumes $x < 2y$, otherwise m and M' do not exist. If $x \geqslant 2y$, let $N \geqslant 0$ be an integer such that $x < 2^{N+1}y$. By applying 16 N times, we can construct a point C' of coordinates $(2^N y, 0)$. The construction above applies, with C' instead of C, and yields a point of coordinates $\left(\frac{x^2}{2^N y}, 0\right)$. Using again 16 N times, we obtain a point of coordinates $\left(\frac{x^2}{y}, 0\right)$.

23 Quotient of product

Initial data:

- a point of coordinates $(0, 0)$;
- a point of coordinates $(x, 0)$;
- a point of coordinates $(y, 0)$;
- a point of coordinates $(z, 0)$.

Constructs: a point of coordinates $(\frac{xy}{z}, 0)$.

Uses: 20, 22.

Using 20, construct a point of coordinates $(\sqrt{xy}, 0)$. Then use 22.

Proof. Obvious. □

24 Multiplication

Initial data:

- a point of coordinates $(0, 0)$;
- a point of coordinates $(1, 0)$;
- a point of coordinates $(x, 0)$;
- a point of coordinates $(y, 0)$.

Constructs: a point of coordinates $(xy, 0)$.

Uses: 23.

Use 23 with $z = 1$.

Proof. Obvious. □

25 Division

Initial data:

- a point of coordinates $(0, 0)$;
- a point of coordinates $(1, 0)$;
- a point of coordinates $(x, 0)$;
- a point of coordinates $(z, 0)$.

Constructs: a point of coordinates $(\frac{x}{z}, 0)$.

Uses: 23.

Use 23 with $y = 1$.

Proof. Obvious. □

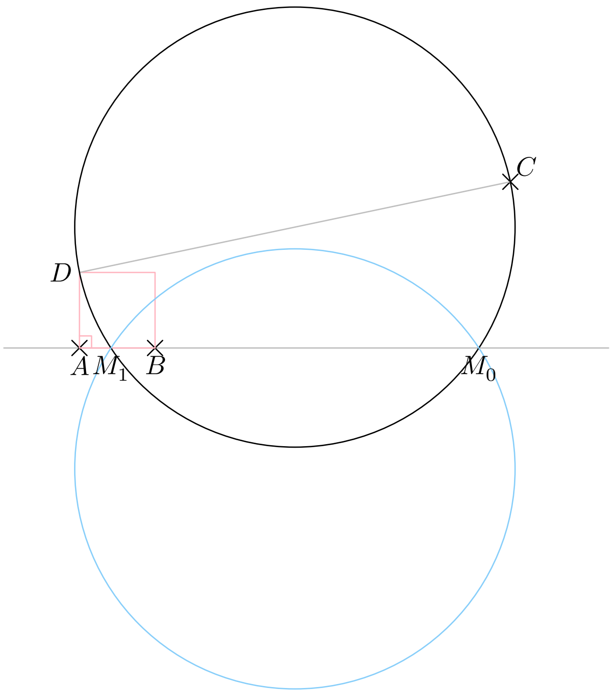
26 Carlyle circle

Initial data:

- a point A of coordinates $(0, 0)$;
- a point B of coordinates $(1, 0)$;
- a point C of coordinates (s, p) such that $s^2 - 4p > 0$.

Constructs: the points of coordinates $(x_0, 0)$ and $(x_1, 0)$, where x_0 and x_1 are the roots of the polynomial $X^2 - sX + p$.

Uses: 6, 9, 19



Using 9, construct the point D of coordinates $(0, 1)$. Using 19, construct the circle of diameter $[CD]$. Using 6, construct the intersections of this cricle with the line (AB). These intersections have coordinates $(x_0, 0)$ and $(x_1, 0)$.

Proof. Left as an exercise. □

In order to use 6, we must ensure that the centre of the circle does not lie on (AB). This is equivalent to $p \neq -1$. If $p = -1$, we can use the doubling construction 16 to solve instead $X^2 - 2sX + 4p$, then use 10 to halve to roots.